Robust model predictive control of linear systems with constraints

Shuyou Yu, Yang Guo, Yu Zhou and Hong Chen

Abstract— In this paper, disturbance observer based model predictive control of linear systems which satisfies a matching condition is proposed, where the disturbance is bounded and varying slowly. A conventional nominal model predictive control problem with tightened constraints is solved online which predicts the nominal trajectory. Two ancillary control laws are determined off-line: one drives the trajectories of the real system to the trajectories of the nominal system, the other tries to cancel out the effect of the disturbance input. Both recursive feasibility of the involved optimization problem and robust stability of systems under control are guaranteed if the optimization problem is feasible at the initial time instant. The resultant online algorithm has similar complexity to that required in conventional model predictive control.

I. INTRODUCTION

Model predictive control (MPC) is one of the most effective techniques available for the control of constrained systems. At each time instant, an optimization problem is solved with the measurement of the system state, and a control sequence is obtained accordingly so as to predict the systems dynamics in a time horizon. However, only is the first segment of the sequence applied to the system. At the next time instant, the whole procedure is repeated with the updated measurement of the system state [1,2].

While dealing with constrained systems with uncertainties, an MPC algorithm is required to ensure the satisfaction of all constraints at all times as well as to guarantee the desired performance. It is natural to consider the worst-case or min-max approach, in which a worst-case performance is minimized online to obtain an optimal control action, suppose that there is no other information of uncertainties except for their bound [3-5]. In general, min-max MPC is very conservative since the worst-case scenario has to be considered at each time instant. On one hand, the worstcase scenario may be not happen at all. On the other hand, the computational burden is very heavy. A great deal of effort is devoted to reduce the computational burden and the conservativeness of the min-max MPC. For linear systems with parameter perturbations or parameter uncertainties, minmax cost function is replaced by its upper bound in [6–9], where a cost function which is upper bound of the min-max cost function is minimized, and a linear feedback control law instead of a sequence of control action is adopted in order to predict the dynamics of the systems. The involved optimization problem solved online can be reduced to a semidefinite programming problem. The idea is easily extended

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to linear parameter varying (LPV) systems [10, 11]. For systems with exogenous disturbances, tube MPC or MPC with tightened constraints are introduced in [12–15], where the control action consists of a nominal control action and a control law. The control action which is obtained by solving online a nominal optimization problem drives the systems dynamics of the nominal systems to the equilibrium. The control law which is obtained offline drives the actual system dynamics to the nominal system dynamics. Tube MPC has almost the same computational burden compared with the nominal MPC, but it only fits for restricted class systems such as linear systems [12, 13] or Lipschitz nonlinear systems [14, 15]. Furthermore, it cannot achieve satisfying effects in controlling systems in the presence of strong disturbances.

Disturbance observer based control can handle the disturbances directly in the process of controller design, rather than asymptotically suppress disturbances through feedback regulation [16–18]. Compound control schemes combining a feedforward compensation part based on disturbance observer and a feedback regulation part based on MPC are addressed to improve disturbance rejection performance of the systems under control [19, 20], where control of both ball mill grinding circuits and dead-time processes is considered, respectively. Robust MPC with a disturbance observer for three-phase voltage source PWM rectifier is presented in [21]. The proposed method has an inherent rapid dynamic response as a result of the conventional MPC. Offset-free MPC is proposed in [22] where the objective of offsetfree is achieved by synthesizing an observer for the nominal systems. Robust MPC for multivariate ill-conditioned systems is addressed in [23], which shows that the optimal disturbance model is close to the input disturbance model. An explicit nonlinear MPC and disturbance observer based control for trajectory tracking of autonomous helicopters is introduced in [24], which provides an effective way of integrating disturbance information.

In this paper, robust MPC of linear systems satisfying a matching condition is considered, where the disturbances are varying slowly and the bound of disturbances could be large. A linear disturbance observer is designed offline to eliminate or cancel off the influence of disturbances. The same as tube MPC, only is a nominal optimization problem solved online. Thus, the proposed scheme has a mild computational burden in general. Both the recursive feasibility of the optimization problem and the ultimate bound of the systems under control are guaranteed suppose that the optimization problem is feasible at the initial time instant. While either the disturbances or derivative of the disturbances are decaying, the system dynamics will approach to the equilibrium.

This paper is organized as follows: In Section II problem setup, basic facts on the adopted disturbance observer and set operation are introduced. The proposed scheme, robust model predictive control based on disturbance observer, is reported in Section III. The properties, both recursive feasibility and ultimate boundedness, are discussed in Section IV. A numerical example is shown in Section V. The paper is concluded with a short summary.

II. PROBLEM SETUP AND PRELIMINARY

Consider linear time-invariant systems

$$\dot{x} = Ax + Bu + B_w w, \tag{1}$$

where $x \in R^{n_x}$ is the system state, $u \in R^{n_u}$ the control input. The term $w \in R^{n_w}$ refers to exogenous disturbance or mismatch between systems and their models. Suppose that x(t) can be measured in real time.

The system is subject to state and input constraints

$$x(t) \in \mathbb{X}, \quad u(t) \in \mathbb{U}, \qquad \forall t > 0,$$
 (2)

where $\mathbb{U} \subseteq \mathbb{R}^{n_u}$ is compact, $\mathbb{X} \subseteq \mathbb{R}^{n_x}$ is connected and (0,0) is contained in the interior of $\mathbb{X} \times \mathbb{U}$.

In order to guarantee stability of closed-loop systems, the following assumptions are made:

Assumption 1: The disturbance $w(t) \in \mathbb{W}$ is bounded, where \mathbb{W} is a bounded set and $0 \in \mathbb{W}$. Furthermore, $\dot{w}(t)$ is piecewise continuous and $\|\dot{w}(t)\| \leq \beta$ for all $t \geq 0$, where β is constant.

Assumption 2: (A, B) is stabilizable. Assumption 3:

$$R(B_w) \subseteq R(B)$$

where R(M) denotes the range (column) space of M. The condition $R(B_w) \subseteq R(B)$ is the so-called matching condition which implies that rank $[B] = \operatorname{rank} \begin{bmatrix} B & B_w \end{bmatrix}$. Assumption 2 together with Assumption 3 describes the structure properties of the considered systems.

The following disturbance observer [16, 17] is designed to estimate the disturbance w

$$\begin{cases} \dot{p} = -LB_w(p + Lx) - L(Ax + Bu),\\ \dot{w} = p + Lx, \end{cases}$$
(3)

where \hat{w} is an estimate of the disturbance, p is an auxiliary variable, and L is the disturbance observer gain to be designed.

Define a nominal system

$$\dot{\bar{x}} := A\bar{x} + B\bar{u},\tag{4}$$

i.e., $w(t) \equiv 0$, $\bar{x}(t) \in \mathbb{X}$ and $\bar{u}(t) \in \mathbb{U}$ for all $t \geq 0$.

A composite linear control law which takes full advantage of the information of the system state x and the estimate of the disturbance w is proposed

$$u = \bar{u} + K(x - \bar{x}) + K_w \hat{w} \tag{5}$$

where K is a state feedback control gain, K_w is the disturbance compensation gain. The aim of designing K_w is to

eliminate or reduce the influence of the disturbances on the system.

Denote $z(t) := x(t) - \bar{x}(t)$ as the error (deviation) between the actual system (1) and the nominal system (4). The dynamics of the *error system* is given as

$$\dot{z} = Az + B\tilde{u} + B_w w \tag{6a}$$

$$\tilde{u} = Kz + K_w \hat{w} \tag{6b}$$

The main objective of this paper is to find a control action $\bar{u}(t)$, disturbance observer gain L, and control feedback laws Kz and $K_w \hat{w}$ for constrained linear systems with respect to disturbances, such that the systems under control is asymptotically ultimately bounded, and constraints (2) are satisfied for all $t \ge 0$ as well.

A. Properties of disturbance observers

The disturbance estimation error of the disturbance observer (3) is defined as

$$e = \hat{w} - w.$$

Combining system (1) and disturbance observer (3), the dynamics of the disturbance estimation error is

$$\dot{e} = -LB_w e - \dot{w}.\tag{7}$$

The ultimate bounded property of the disturbance observer is concluded by the following lemma.

Lemma 1: [18] Suppose that Assumption 1 and Assumption 2 are satisfied for system (1). Then, there exist $M \ge 1$ and $\gamma < 0$ such that the disturbance estimate $\hat{w}(t)$ yielded by the disturbance observer (3) will asymptotically track the disturbances w(t) with the ultimate bound error $-\frac{\beta M}{\gamma}$ if the observer gain L is chosen such that $-LB_w$ is Hurwitz. Note that the disturbance estimates \hat{w} yielded by the disturbance observer (3) will asymptotically track the disturbance observer (3) w

bance observer (3) will asymptotically track the disturbance w offset if the observer gain matrix L is chosen such that $-LB_w$ is Hurwitz and $\lim_{t\to\infty} \dot{w}(t) = 0$.

Combining system (1), the composite control law (5) and the dynamics of the disturbance estimation error (7), the closed-loop error system is

$$\begin{bmatrix} \dot{z} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A + BK & BK_w \\ 0 & -LB_w \end{bmatrix} \begin{bmatrix} z \\ e \end{bmatrix} + \begin{bmatrix} BK_w + B_w & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} w \\ \dot{w} \end{bmatrix}$$
(8)

In terms of Assumption 3, there exists a K_w such that $BK_w + B_w = 0$.

Lemma 2: [18] Suppose that Assumption 1-3 are satisfied for system (6). If K, L and K_w are chosen such that

(1) both
$$-LB_w$$
 and $A + BK$ are Hurwitz,

(2)
$$BK_w + B_w = 0.$$

Then,

- (a) system (6) under the feedback control law $Kz + K_w \hat{w}$ is input-to-state stable (ISS) from \dot{w} to state z,
- (b) system state z(t) is bounded which is proportional to $\|\dot{w}(t)\|_{\infty}$.

Note that $\lim_{t\to\infty} z(t) = 0$, i.e., the effect of the disturbance is eliminated as t goes to infinity, if $\lim_{t\to\infty} \dot{w}(t) = 0$.

B. Set operations

Before proceeding the main results of the paper it is necessary to introduce some set operations [25].

Definition 1: Consider two sets $\mathcal{A}, \mathcal{B} \subset \mathbb{R}^n$, then the Pontryagin difference set is defined as

$$\mathcal{A} \ominus \mathcal{B} = \{ x \in \mathbb{R}^n | x + y \in \mathcal{A}, \forall y \in \mathcal{B} \},\$$

Similarly, the addition set is defined as

 $\mathcal{A} \oplus \mathcal{B} = \{ x + y | x \in \mathcal{A}, x \in \mathcal{B}, \}.$

Definition 2: The multiplication of a set \mathcal{B} by a matrix A denotes a mapping of all its elements

$$A\mathcal{B} = \{c \mid \exists b \in \mathcal{B}, c = Ab\},\$$

III. ROBUST MODEL PREDICTIVE CONTROL

In this section robust model predictive control of linear systems with disturbances is proposed. The controller has two components in which a nominal controller generates a nominal state trajectory, the ancillary control laws K and K_w aims at steering the trajectories of the perturbed system (6) to the nominal one, and cancelling out the effect of disturbances, respectively.

A. Nominal control input

The optimization problem solved online is subject to the nominal dynamics (4), i.e. no disturbances are presented. Furthermore, tightened constraints rather than the original constraints are introduced in order to guarantee the satisfaction of the input constraints and state constraints.

For the system state $x(t_k)$, the optimization problem solved online is formulated as follows:

Problem 1:

$$\min_{\bar{u}(\cdot,\bar{x}(t_k))} \quad J\left(\bar{x}(t_k), \bar{u}(\cdot,\bar{x}(t_k))\right)$$
(9a)

subject to

$$\dot{\bar{x}}(t) = A\bar{x}(t) + B\bar{u}(t), \tag{9b}$$

$$\bar{\bar{x}}(t_{1} + \tau; \bar{\bar{x}}(t_{1})) \in \mathbb{N}_{0}, \quad \tau \in [t_{1}, t_{1} + T] \tag{9c}$$

$$\bar{x}(t_k + \tau; \bar{x}(t_k)) \in \mathbb{X}_0, \qquad \tau \in [t_k, t_k + T_p], \qquad (9c)$$
$$\bar{u}(t_k + \tau; \bar{x}(t_k)) \in \mathbb{H}_0, \qquad \tau \in [t_k, t_k + T_k] \qquad (9d)$$

$$\bar{u}(t_k + \tau; \bar{x}(t_k)) \in \mathbb{U}_0, \qquad \tau \in [t_k, t_k + T_p], \qquad (9d)$$
$$\bar{u}(t_k + T; \bar{x}(t_k)) \in \mathbb{X}$$

$$\bar{x}(t_k + T_p; \bar{x}(t_k)) \in \mathbb{X}_f,$$
(9e)

where $\mathbb{X}_0 := \mathbb{X} \ominus \Omega$, $\mathbb{U}_0 := \mathbb{U} \ominus (K\Omega \oplus K_w \mathbb{W})$, $\mathbb{X}_f \subseteq \mathbb{X} \ominus \Omega$. The set Ω is the ultimate bound of the error z which will be introduced in detail later.

The objective function is

$$J(\bar{x}(t_{k}), \bar{u}(\cdot)) := E(\bar{x}(t_{k} + T_{p}; \bar{x}(t_{k}))) + \int_{t_{k}}^{t_{k} + T_{p}} \bar{x}^{T}(\tau; \bar{x}(t_{k}))Q\bar{x}(\tau; \bar{x}(t_{k})) + \bar{u}^{T}(\tau; \bar{x}(t_{k}))R\bar{u}(\tau; \bar{x}(t_{k}))d\tau$$

where T_p is the prediction horizon, $Q \in \mathbb{R}^{n \times n}$ and $R \in \mathbb{R}^{m \times n}$ are positive definite weighting matrices. The term $\bar{u}(\cdot, \bar{x}(t_k))$ denotes the optimal solution to Problem1, and $\bar{x}^*(\cdot; \bar{x}^*(t_k))$ is the predicted trajectory of the nominal system (4) starting from the state $\bar{x}^*(t_k)$ at time t_k and driven by the optimal control input $\bar{u}^*(\cdot; \bar{x}^*(t_k))$.

Denote δ as the sampling instant, i.e., the optimization problem is solved at each $k\delta$ with $k \in [1, 2, 3, \cdots)$. The control input of the nominal system (4) during the sampling interval $[t_k, t_k + \delta)$ is

$$\bar{u}(\tau) = \bar{u}^*(\tau; \bar{x}^*(t_k)).$$
 (10)

The terminal set X_f is a neighborhood of the origin which is a sub-level set of a positive definite function $E(\cdot)$. Moreover, X_f and E(x) satisfy the following conditions [2, 26]:

B0) $X_f \subseteq \mathbb{X}_0$,

B1) $\bar{u} \in \mathbb{U}_0$, for all $\bar{x} \in \mathbb{X}_f$,

B2) For all $\bar{x} \in \mathbb{X}_f$, $E(\bar{x})$ satisfies

$$\frac{\partial E(\bar{x})}{\partial \bar{x}} (A\bar{x} + B\bar{u}) + \bar{x}^T Q\bar{x} + \bar{u}^T R\bar{u} \le 0.$$
(11)

Since (A, B) is staiblizable, there exists a locally asymptotically stabilizing control law $\bar{u} := K\bar{x}$, a terminal region $X_f := \{\bar{x} \in R^{n_x \times n_x} \mid \bar{x}^T P \bar{x} \leq \alpha\}$ with P positive definite and $\alpha > 0$, a positive definite function $E(\bar{x}) :=$ $\bar{x}^T P \bar{x}$ that satisfy (11) for all $\bar{x} \in \mathbb{X}_f$ [2]. Since (11) holds, X_f is invariant for the nominal system (4) controlled by $\bar{u} = K\bar{x}$. Model predictive control based on repeated solution of Problem 1 stabilizes the nominal system if the terminal conditions B0-B2 are satisfied [1, 2]. Furthermore, the optimal value function J^* satisfies:

$$J(\bar{x}^{*}(t+\delta;\bar{x}(t))) - J^{*}(\bar{x}(t)) \leq \int_{t}^{t+\delta} \bar{x}^{*}(\tau)^{T} Q \bar{x}^{*}(\tau) - \bar{u}^{*}(\tau)^{T} R \bar{u}^{*}(\tau) d\tau.$$

B. Ultimate boundedness

Definition 3: A system is asymptotically ultimately bounded if the system converges asymptotically to a bounded set.

In terms of $BK_w + B_w = 0$, Eq.(8) is rewritten as

$$\begin{bmatrix} \dot{z} \\ \dot{e} \end{bmatrix} = \underbrace{\begin{bmatrix} A + BK & BK_w \\ 0 & -LB_w \end{bmatrix}}_{\tilde{A}} \begin{bmatrix} z \\ e \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ -1 \end{bmatrix}}_{\tilde{B}} \dot{w}$$
(12a)

$$z = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\tilde{C}} \begin{bmatrix} z \\ e \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{\tilde{D}} \dot{w}.$$
 (12b)

The following result shows that there exists an ultimate bound for the augmented system (12).

Lemma 3: [27] Suppose that z(0) = 0, and there exists R > 0, $\lambda > 0$, $\gamma > 0$ and $\mu > 0$ such that

$$\begin{bmatrix} \tilde{A}^T R + R\tilde{A} + \lambda R & R\tilde{B} \\ \tilde{B}^T R & -\mu I \end{bmatrix} < 0,$$
(13a)

$$\begin{bmatrix} \lambda R & 0 & \tilde{C}^T \\ 0 & (\gamma - \mu)I & \tilde{D}^T \\ \tilde{C} & \tilde{D} & \gamma I \end{bmatrix} > 0,$$
 (13b)

then $z(t)^T z(t) \leq \gamma^2 \dot{w}(t)^T \dot{w}(t)$ for all $t \geq 0$.

Theorem 1: Suppose that z(0) = 0, and there exist positive definite matrices $X_1 \in \mathcal{R}^{n_x \times n_x}$ and $X_2 \in \mathcal{R}^{n_w \times n_w}$, non-square matrices $Y_1 \in \mathcal{R}^{n_u \times n_x}$ and $Y_2 \in \mathcal{R}^{n_w \times n_x}$, and scalars $\lambda > 0$, $\gamma > 0$ and $\mu > 0$ such that

$$\begin{bmatrix} \Pi & BK_w & 0\\ K_w^T B^T & \lambda X_2 - Y_2 B_w - B_w^T Y_2^T & -X_2\\ 0 & -X_2 & -\mu I \end{bmatrix} < 0 \quad (14a)$$
$$\begin{bmatrix} \lambda X_1 & X_1\\ X_1 & \gamma I \end{bmatrix} > 0 \quad (14b)$$

with $\prod := X_1 A^T + A X_1 + Y_1^T B^T + B Y_1 + \lambda X_1$. Then, with $K := Y_1 X_1^{-1}$ and $L := Y_2 X_2^{-1}$, the error system (12) with the control law $\tilde{u} = Kz + K_w \hat{w}$ is ultimate bounded.

Proof: The proof is divided into three parts. First Eq.(14a) and Eq.(14b) are satisfied for the system (12), respectively. Then, based on Lemma 3, the upper bound of the error system is obtained.

(1) Choosing $R = \begin{bmatrix} H_1 & 0 \\ 0 & X_2 \end{bmatrix}$ with $H_1 > 0$, and taking $\tilde{A} = \begin{bmatrix} A + BK & BK_w \\ 0 & -LB_w \end{bmatrix}$ and $\tilde{B} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ into account, (13a) can be rewritten

$$\begin{bmatrix} \Pi_{1} & H_{1}BK_{w} & 0\\ K_{w}^{T}B^{T}H_{1} & \lambda X_{2} - X_{2}LB_{w} - B_{w}^{T}L^{T}X_{2} & -X_{2}\\ 0 & -X_{2} & -\mu I \end{bmatrix} < 0$$
(15)

with $\prod_{1} := (A + BK)^{T} H_{1} + H_{1}(A + BK) + \lambda H_{1}.$

Denote $Y_2 = X_2L$ and perform a congruence transformation with diag $\{H_1^{-1}, I, I\}$, we obtain that

$$\begin{bmatrix} \prod_{1} & BK_{w} & 0\\ K_{w}^{T}B^{T} & \lambda X_{2} - Y_{2}B_{w} - B_{w}^{T}Y_{2}^{T} & -X_{2}\\ 0 & -X_{2} & -\mu I \end{bmatrix} < 0$$
with
$$\prod_{1} := H_{1}^{-1}A^{T} + H_{1}^{-1}K^{T}B^{T} + AH_{1}^{-1} + BKH_{1}^{-1} - \mu I = 0$$

Furthermore, denote $X_1 := H_1^{-1}$ and $Y_1 := KX_1$, the

above equation is equivalent to (14a). (2) Taking $\tilde{D} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\tilde{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$ into account, (13b) can be rewritten as

$$\begin{bmatrix} \lambda H_1 & 0 & I \\ 0 & X_2 & 0 \\ I & 0 & \gamma I \end{bmatrix} > 0.$$

In terms of $X_2 > 0$, the above equation is simplified as

$$\begin{bmatrix} \lambda H_1 & I\\ I & \gamma I \end{bmatrix} > 0. \tag{16}$$

Performing a congruence transformation with diag{ H_1^{-1} , I}, (14b) is obtained.

(3) Due to Lemma 3, the upper bound of z(t) for all $t \ge 0$ is obtained, i.e., $||z||_{\infty} \leq \gamma \beta$.

Remark 3.1: Since $\prod := X_1 A^T + A X_1 + Y_1^T B^T + B Y_1 +$ $\lambda X_1 > 0$, the system $\dot{z}(t) = Az(t) + Bu(t)$ with u = $Y_1 X_1^{-1} z$ is exponentially stable. Thus, the initial state z(0) of the systems (12) has little influence on the system dynamics.

Remark 3.2: Lemma 2 shows that z(t) is bounded for all $t \ge 0$, and Theorem 2 estimates the upper bound of z(t). Algorithm 1: (Offline)

- (1) Choose K_w such that $BK_w + B_w = 0$.
- (2) Obtain L and K by solving the matrix inequalities (14a) and (14b).

Then, the upper bound of the error system is

$$\Omega := \{ z \in \mathbb{R}^{n_x} \mid \|z\|_{\infty} \le \gamma\beta \}$$

The following online algorithm is implemented in this paper after (K_w, K, L) is chosen offline.

Algorithm 2: (Online)

Date: (K_w, K, L)

Initialization: $x(t_0)$, $w(t_0)$ and $p(t_0)$

Step 0: At time t_0 , set $\bar{x}(t_0) = x(t_0)$ where $x(t_0)$ is the current state.

Step 1: At time t_k , solve Problem 1 with the current state $\bar{x}(t_k)$ to obtain the nominal control action $\bar{u}(t_k)$, compute the successor state $p(t_{k+1})$ and $\hat{w}(t_{k+1})$ of the disturbance observer (3), and the actual control action $u(t_k) = \bar{u}(t_k) + K(x(t_k) - \bar{x}(t_k)) + K_w \hat{w}.$

Step 2: Apply the control $u(t_k)$ to the system (1) during the sampling interval $[t_k, t_{k+1}]$, where $t_{k+1} = t_k + \delta$. **Step 3:** Measure the state $x(t_{k+1})$ at the next time instant t_{k+1} of the system (1), compute the successor state $\bar{x}(t_{k+1})$ of the nominal system (4) under the nominal control $\bar{u}(t_k)$.

Step 4: Set $(\bar{x}(t_k), x(t_k)) = (\bar{x}(t_{k+1}), x(t_{k+1}))$ and $(p(t_k), \hat{w}(t_k)) = (p(t_{k+1}), \hat{w}(t_{k+1}),), t_k = t_{k+1}, \text{ and}$ go to Step 1.

IV. PROPERTIES OF THE PROPOSED SCHEME

The proposed scheme has the same online computational burden as the conventional MPC with guaranteed nominal stability [1, 2] since only the nominal model is used to predict the system dynamics.

The properties of the systems with the proposed scheme are stated in the following theorem.

Theorem 2: Suppose that Problem 1 is feasible at the initial time t_0 . Then,

- (i) Problem 1 is feasible for all $t > t_0$,
- (ii) the system state x(t) converges to Ω , i.e., the system under control is ultimately bounded.

Proof: (1) Only the nominal state and the nominal system dynamics are used while Problem 1 is solved at each time instant, the solution of the optimization problem does not depend on the external disturbances at all. Therefore, recursive feasibility is guaranteed suppose that Problem 1 is feasible at the initial time instant [2].

(2) Since the nominal system (4) with model predictive control is asymptotic stable [1,2], there exists a class \mathcal{KL} function $\beta(\bar{x}, t)$ [28] such that

$$\|\bar{x}(t)\| \le \beta(\bar{x}(t_0), t), \quad \forall t \ge t_0.$$

In terms of Theorem 1, $z(t) \leq \gamma \beta$ for all $t \geq t_0$. Thus,

 $\|x(t)\| \le \beta(\bar{x}(t_0), t) + \gamma\beta, \quad \forall t \ge t_0,$

since $x(t) = \bar{x}(t) + z(t)$ for all $t \ge t_0$.

For a given $\varepsilon > 0$, there exists t_{ε} such that for all $t \ge t_{\varepsilon}$

$$\|\bar{x}(t) - 0\| = \|\bar{x}(t)\| \le \beta(\bar{x}(t_0), t) \le \varepsilon,$$

i.e., $\lim_{t\to\infty} \bar{x}(t) = \lim_{t\to\infty} \left[x(t) - z(t) \right] = 0.$ Thus,

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} z(t) = \gamma\beta$$

V. ILLUSTRATIVE EXAMPLES

The system is

$$\dot{x} = Ax + Bu + B_w w \tag{17}$$

with $A = \begin{bmatrix} -1 & 2 \\ -3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0.5 & -2 \\ -5 & 0.5 \end{bmatrix}$, $B_w = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$, where the matrix A has two eigenvalues +2.2361*i* and -2.2361*i*, and the matching condition $R(B_w) \subseteq R(B)$ is satisfied. Assume that $x_1(t)$ and $x_2(t)$ can be measured, and the control constraint is

$$-2 \le u_i(t) \le 2, \quad \forall t \ge 0, \quad i = 1, 2.$$

For all $t \ge 0$, the disturbance satisfies that $w(t) \in \mathbb{W}$ with

$$\mathbb{W} := \left\{ w \in R^1 \mid |w| \le 0.5, |\dot{w}| \le 0.05 \right\}.$$

The nominal system

$$\dot{\bar{x}} = A\bar{x} + B\bar{u}$$

is controllable. The weighting matrices of Problem 1 are chosen as

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Choose $K_w = \begin{bmatrix} -0.0256\\ -0.2564 \end{bmatrix}$ such that $BK_w + B_w = 0$. Fix $\lambda = 1$, and solve matrix inequalities (14a) and (14a) to obtain $\gamma = 3.2466$, $K = \begin{bmatrix} 0.5609 & 0.2641\\ 0.2641 & -1.7702 \end{bmatrix}$ and $L = \begin{bmatrix} 2.8672\\ 0.0000 \end{bmatrix}$. Denote K_1 and K_2 as the first and second row of the linear

Denote K_1 and K_2 as the first and second row of the linear control gain K, i.e., $K_1 = \begin{bmatrix} 0.5609 & 0.2641 \end{bmatrix}$ and $K_2 = \begin{bmatrix} 0.2641 & -1.7702 \end{bmatrix}$. In terms of $||z||_{\infty} \le \gamma\beta = 0.1623$, $||K_1z||_{\infty} \le 0.1339$, and $||K_2z||_{\infty} \le 0.3302$. Suppose that $||\hat{w}(t)|| \le 0.5$ for all $t \ge 0$, i.e., an output saturation function is added for the designed disturbance observer (3). Denote K_{w1} and K_{w2} as the first and second row of the linear control gain K_w , i.e., $K_{w1} = \begin{bmatrix} -0.0256 \end{bmatrix}$ and $K_{w2} = \begin{bmatrix} -0.2564 \end{bmatrix}$. Thus, $||K_{w1}\hat{w}||_{\infty} \le 0.0128$, and $||K_{w2}\hat{w}||_{\infty} \le 0.1282$. Therefore, the control input of Problem 1 is

$$-1.85 \le u_1(t) \le 1.85, \\ -1.50 \le u_2(t) \le 1.50,$$

for all $t \ge 0$.

Both the terminal control law and the terminal penalty matrices are computed by the solution of a convex optimization problem, see [29], the terminal set $\mathcal{X}_f = \{\bar{x} \in \mathbb{R}^2 \mid E(\bar{x}) \leq 10\}$ with $E(\bar{x}) = \bar{x}^T \begin{bmatrix} 0.4103 & -0.0641 \\ -0.0641 & 0.4180 \end{bmatrix} \bar{x}$ and the terminal control law $\pi(\bar{x}) = \begin{bmatrix} -0.0439 & 0.4083 \\ 0.3973 & -0.0207 \end{bmatrix} \bar{x}$.

Problem 1 is solved in discrete time with a sampling time of $\delta = 0.02$ time units and a prediction horizon of $T_p = 0.2$ time units. Figure 1 shows the state trajectory of the considered system starting at $x_0 = [-2.5-2.5]^T$ with respect to the disturbance

$$w(t) = \begin{cases} 0 & t \in [0,4), \\ 0.5(1 - e^{-0.01(t-4)}) & t \in [4,+\infty). \end{cases}$$

The inputs injected to the real system satisfy the constraint although the potential of the inputs is not taken full advantage of. The estimated disturbance can converge to the actual disturbance in finite time. The system state converges to the equilibrium even if the non-zero disturbances exist as $\lim_{t\to\infty} \dot{w}(t) = 0.$

VI. CONCLUSIONS

In this paper, a robust model predictive control scheme of linear systems was proposed, where the considered disturbances are varying slowly and the systems satisfy a matching condition. A nominal optimization problem was solved online to predict the dynamics of the nominal systems, and disturbance observer was designed in order to eliminate the influence of the disturbances. A linear control law drives the system dynamics of the real system to the system dynamics of the nominal systems. Both the recursive feasibility of the involved optimization problem and the asymptotic stability of the systems under control are guaranteed if the optimization problem has a feasible solution at the initial time instant. The proposed scheme has the same computational burden as the conventional model predictive control. The effectiveness of the proposed scheme is verified by a simulation example.

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Fig. 1. Exemplary time profiles for the closed-loop system (17) for disturbances w from $x_0 = \begin{bmatrix} -2.5 & -2.5 \end{bmatrix}^T$. For the figure of disturbance w, the solid line: estimated disturbances, dashed line: actual disturbances

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